

CONSTRUCTING SIMPLICIAL BRANCHED COVERS

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ABSTRACT. Izmistiev and Joswig described how to obtain a simplicial covering space (the *partial unfolding*) of a given simplicial complex, thus obtaining a simplicial branched cover [Adv. Geom. 3(2):191-255, 2003]. We present a large class of branched covers which can be constructed via the partial unfolding. In particular, for $d \leq 4$ every closed oriented PL d -manifold is the partial unfolding of some polytopal d -sphere.

1. INTRODUCTION

Branched covers are applied frequently in topology – most prominently in the study, construction and classification of closed oriented PL d -manifolds. First results are by Alexander [1] in 1920, who observed that any closed oriented PL d -manifold M is a branched cover of the d -sphere. Unfortunately Alexander’s proof does not allow for any (reasonable) control over the number of sheets of the branched cover, nor over the topology of the branching set: The number of sheets depends on the size of some triangulation of M and the branching set is the co-dimension 2-skeleton of the d -simplex.

However, in dimension $d \leq 4$, the situation is fairly well understood. By results of Hilden [8] and Montesinos [17] any closed oriented 3-manifold M arises as 3-fold simple branched cover of the 3-sphere branched over a link. In dimension four the situation becomes increasingly difficult. First Piergallini [21] showed how to obtain any closed oriented PL 4-manifold as a 4-fold branched cover of the 4-sphere branched over a transversally immersed PL-surface [21]. Iori & Piergallini [11] then improved the standing result showing that the branching set may be realized locally flat if one allows for a fifth sheet for the branched cover, thus proving a long-standing conjecture by Montesinos [18]. The question as to whether any closed oriented PL 4-manifold can be obtained as 4-fold cover of the 4-sphere branched over a locally flat PL-surface is still open.

For the partial unfolding and the construction of closed oriented combinatorial 3-manifolds we recommend Izmistiev & Joswig [14]. Their construction has recently been simplified significantly by Hilden, Montesinos-Amilibia, Tejada & Toro [9]. For those able to read German additional analysis and examples can be found in [24]. The partial unfolding is implemented in the software package `polymake` [6].

This work has been greatly inspired by a paper of Hilden, Montesinos-Amilibia, Tejada & Toro [9] and their bold approach. However, the techniques developed in the following turn out to differ substantially from the ideas in [9], allowing for stronger results in dimension three and generalization to arbitrary dimensions.

Outline of the paper. After some basic definitions and notations the partial unfolding \widehat{K} of a simplicial complex K is introduced. The partial unfolding defines a projection $p : \widehat{K} \rightarrow K$ which is a simplicial branched cover if K meets certain connectivity assumptions. We define combinatorial models of key features of a branched cover, namely the branching set and the monodromy homomorphism.

Sections 2 and 3 are related, yet self contained. The main result of this paper is presented in Theorem 2.1 and we give an explicit construction of a combinatorial d -sphere S , such that

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$p : \widehat{S} \rightarrow S$ is equivalent to a given simple, $(d+1)$ -fold branched cover $r : X \rightarrow \mathbb{S}^d$ (with some additional restriction for the branching set of r). Theorem 2.1 is then applied to the construction of closed oriented PL d -manifolds as branched covers for $d \leq 4$. The construction of S and the proof of its correctness take up the entire Section 2.

Finally, in Section 3 we discuss how to extend k -coloring of a subcomplex $L \subset K$ of a simplicial d -complex K to a $\max\{k, d+1\}$ -coloring of a refinement K' of K , such that L is again a subcomplex of K' . Since K' is constructed from K via finitely many stellar subdivisions of edges, all properties invariant under these subdivisions are preserved, e.g. polytopality, regularity, shellability, and others. This improves an earlier result by Izmistiev [12].

1.1. Basic definitions and notations. A simplicial complex K is a *combinatorial d -sphere* or *combinatorial d -ball* if it is piecewise linear homeomorphic to the boundary of the $(d+1)$ -simplex, respectively to the d -simplex. Equivalently, K is a combinatorial d -sphere or d -ball if there is a common refinement of K and the boundary of the $(d+1)$ -simplex, respectively the d -simplex. A simplicial complex K is a *combinatorial manifold* if the vertex link of each vertex of K is a combinatorial sphere or a combinatorial ball. A manifold M is PL if and only if M has a triangulation as a combinatorial manifold. For an introduction to PL-topology see Björner [2, Part II], Hudson [10], and Rourke & Sanderson [22].

A finite simplicial complex is *pure* if all the inclusion maximal faces, called the *facets*, have the same dimension. We call a co-dimension 1-face of a pure simplicial complex K a *ridge*, and the *dual graph* $\Gamma^*(K)$ of K has the facets as its node set, and two nodes are adjacent if they share a ridge. We denote the 1-skeleton of K by $\Gamma(K)$, its *graph*.

Further it is often necessary to restrict ourselves to simplicial complexes with certain connectivity properties: A pure simplicial complex K is *strongly connected* if its dual graph $\Gamma^*(K)$ is connected, and *locally strongly connected* if the star $\text{st}_K(f)$ of f is strongly connected for each face $f \in K$. If K is locally strongly connected, then connected and strongly connected coincide. Further we call K *locally strongly simply connected* if for each face $f \in K$ with co-dimension ≥ 2 the link $\text{lk}_K(f)$ of f is simply connected, and finally, K is *nice* if it is locally strongly connected and locally strongly simply connected. Observe that combinatorial manifolds are always nice.

Let $(\sigma_0, \sigma_1, \dots, \sigma_l)$ be an ordering of the facets of a pure simplicial d -complex K , and let $D_i = \bigcup_{0 \leq j \leq i} \sigma_j$ denote the union of the first i facets. We call the ordering $(\sigma_0, \sigma_1, \dots, \sigma_l)$ a *shelling* of K if $D_{i-1} \cap \sigma_i$ is a pure simplicial $(d-1)$ -complex for $1 \leq i \leq l$. If K is the boundary complex of a simplicial $(d+1)$ -polytope, then K admits a shelling order which can be computed efficiently; see Ziegler [27, Chapter 8].

A simplicial complex obtained from a shellable complex by stellar subdivision of a face is again shellable, a shellable sphere or ball is a combinatorial sphere or ball, and for $1 \leq i \leq l$ the intersection $D_{i-1} \cap \sigma_i$ is a combinatorial $(d-1)$ -ball (or sphere). A shellable simplicial complex K is a wedge of balls or spheres in general. If K is a manifold, then D_i is a combinatorial d -ball (or sphere) for $0 \leq i \leq l$, and in particular we have that $D_{i-1} \cap \sigma_i$, D_i , and hence K are nice. We call a face $f \subset \sigma_i$ *free* if $f \notin D_{i-1}$. In particular the (inclusion) minimal free faces describe all free faces, and they are also called *restriction sets* in the theory of h -vectors of simplicial polytopes.

1.2. The branched cover. The concept of a covering of a space Y by another space X is generalized by Fox [4] to the notion of the branched cover. Here a certain subset $Y_{\text{sing}} \subset Y$ may violate the conditions of a covering map. This allows for a wider application in the construction of topological spaces. It is essential for a satisfactory theory of (branched) coverings to make certain connectivity assumption for X and Y . The spaces mostly considered are Hausdorff, path connected, and locally path connected; see Bredon [3, III.3.1]. Throughout we will restrict our attention to coverings of manifolds hence they meet the connectivity assumptions in [3].

Consider a continuous map $h : Z \rightarrow Y$, and assume the restriction $h : Z \rightarrow h(Z)$ to be a covering. If $h(Z)$ is dense in Y (and meets certain additional connectivity conditions) then there is a surjective map $p : X \rightarrow Y$ with $Z \subset X$ and $p|_Z = h$. The map p is called a *completion* of h , and any two completions $p : X \rightarrow Y$ and $p' : X' \rightarrow Y$ are equivalent in the sense that there

exists a homeomorphism $\varphi : X \rightarrow X'$ satisfying $p' \circ \varphi = p$ and $\varphi|_Z = \text{Id}$. The map $p : X \rightarrow Y$ obtained this way is a *branched cover*, and we call the unique minimal subset $Y_{\text{sing}} \subset Y$ such that the restriction of p to the preimage of $Y \setminus Y_{\text{sing}}$ is a covering, the *branching set* of p . The restriction of p to $p^{-1}(Y \setminus Y_{\text{sing}})$ is called the *associated covering* of p . If $h : Z \rightarrow Y$ is a covering, then $X = Z$, and $p = h$ is a branched cover with empty branching set.

Example 1.1. For $k \geq 2$ consider the map

$$p_k : \mathbb{C} \rightarrow \mathbb{C} : z \mapsto z^k.$$

The restriction $p_k|_{\mathbb{D}^2}$ is a k -fold branched cover $\mathbb{D}^2 \rightarrow \mathbb{D}^2$ with the single branch point $\{0\}$.

We define the *monodromy homomorphism*

$$\mathfrak{m}_p : \pi_1(Y \setminus Y_{\text{sing}}, y_0) \rightarrow \text{Sym}(p^{-1}(y_0))$$

of a branched cover for a point $y_0 \in Y \setminus Y_{\text{sing}}$ as the monodromy homomorphism of the associated covering: If $[\alpha] \in \pi_1(Y \setminus Y_{\text{sing}}, y_0)$ is represented by a closed path α based at y_0 , then \mathfrak{m}_p maps $[\alpha]$ to the permutation $(x_i \mapsto \alpha_i(1))$, where $\{x_1, x_2, \dots, x_k\} = p^{-1}(y_0)$ is the preimage of y_0 and $\alpha_i : [0, 1] \rightarrow X$ is the unique lifting of α with $p \circ \alpha_i = \alpha$ and $\alpha_i(0) = x_i$; see Munkres [19, Lemma 79.1] and Seifert & Threlfall [23, § 58]. The *monodromy group* \mathfrak{M}_p is defined as the image of \mathfrak{m}_p .

Two branched covers $p : X \rightarrow Y$ and $p' : X' \rightarrow Y'$ are *equivalent* if there are homeomorphisms $\varphi : X \rightarrow X'$ and $\psi : Y \rightarrow Y'$ with $\psi(Y_{\text{sing}}) = Y'_{\text{sing}}$, such that $p' \circ \varphi = \psi \circ p$ holds. The well known Theorem 1.2 is due to the uniqueness of Y_{sing} , and hence the uniqueness of the associated covering; see Piergallini [20, p. 2].

Theorem 1.2. *Let $p : X \rightarrow Y$ be a branched cover of a connected manifold Y . Then p is uniquely determined up to equivalence by the branching set Y_{sing} , and the monodromy homeomorphism \mathfrak{m}_p . In particular, the covering space X is determined up to homeomorphism.*

Let Y be a connected manifold and Y_{sing} a co-dimension 2 submanifold, possibly with a finite number of singularities. We call a branched cover p *simple* if the image $\mathfrak{m}_p(m)$ of any meridial loop m around a non-singular point of the branching set is a transposition in \mathfrak{M}_p . Note that the k -fold branched cover $p_k|_{\mathbb{D}^2} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ presented in Example 1.1 is not simple for $k \geq 3$.

1.3. The partial unfolding. The partial unfolding \hat{K} of a simplicial complex K first appeared in a paper by Izmistiev & Joswig [14], with some of the basic notions already developed in Joswig [15]. The partial unfolding is closely related to the complete unfolding, also defined in [14], but we will not discuss the latter. The partial unfolding is a geometric object defined entirely by the combinatorial structure of K , and comes along with a canonical projection $p : \hat{K} \rightarrow K$.

However, the partial unfolding \hat{K} may not be a simplicial complex. In general \hat{K} is a pseudo-simplicial complex: Let Σ be a collection of pairwise disjoint geometric simplices with simplicial attaching maps for some pairs $(\sigma, \tau) \in \Sigma \times \Sigma$, mapping a subcomplex of σ isomorphically to a subcomplex of τ . Identifying the subcomplexes accordingly yields the quotient space Σ/\sim , which is called a *pseudo-simplicial complex* if the quotient map $\Sigma \rightarrow \Sigma/\sim$ restricted to any $\sigma \in \Sigma$ is bijective. The last condition ensures that there are no self-identifications within each simplex $\sigma \in \Sigma$.

The group of projectivities. Let σ and τ be neighboring facets of a finite, pure simplicial complex K , that is, $\sigma \cap \tau$ is a ridge. Then there is exactly one vertex in σ which is not a vertex of τ and vice versa, hence a natural bijection $\langle \sigma, \tau \rangle$ between the vertex sets of σ and τ is given by

$$\langle \sigma, \tau \rangle : V(\sigma) \rightarrow V(\tau) : v \mapsto \begin{cases} v & \text{if } v \in \sigma \cap \tau \\ \tau \setminus \sigma & \text{if } v = \sigma \setminus \tau. \end{cases}$$

The bijection $\langle \sigma, \tau \rangle$ is called the *perspectivity* from σ to τ .

A *facet path* in K is a sequence $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$ of facets, such that the corresponding nodes in the dual graph $\Gamma^*(K)$ form a path, that is, $\sigma_i \cap \sigma_{i+1}$ is a ridge for all $0 \leq i < k$;

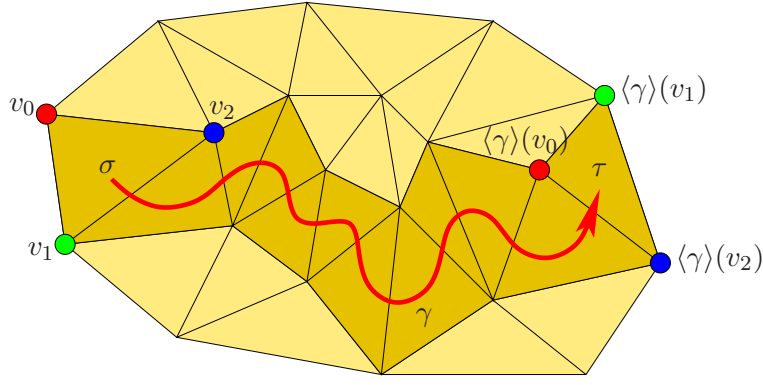


FIGURE 1. A projectivity from σ to τ along the facet path γ .

see Figure 1. Now the *projectivity* $\langle \gamma \rangle$ along γ is defined as the composition of perspectivities $\langle \sigma_i, \sigma_{i+1} \rangle$, thus $\langle \gamma \rangle$ maps $V(\sigma_0)$ to $V(\sigma_k)$ bijectively via

$$\langle \gamma \rangle = \langle \sigma_{k-1}, \sigma_k \rangle \circ \cdots \circ \langle \sigma_1, \sigma_2 \rangle \circ \langle \sigma_0, \sigma_1 \rangle.$$

We write $\gamma \delta = (\sigma_0, \sigma_1, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{k+l})$ for the *concatenation* of two facet paths $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$ and $\delta = (\sigma_k, \sigma_{k+1}, \dots, \sigma_{k+l})$, denote by $\gamma^- = (\sigma_k, \sigma_{k-1}, \dots, \sigma_0)$ the *inverse path* of γ , and we call γ a *closed* facet path based at σ_0 if $\sigma_0 = \sigma_k$. The set of closed facet paths based at σ_0 together with the concatenation form a group, and a closed facet path γ based at σ_0 acts on the set $V(\sigma_0)$ via $\gamma \cdot v = \langle \gamma \rangle(v)$ for $v \in V(\sigma_0)$. Via this action we obtain the *group of projectivities* $\Pi(K, \sigma_0)$ given by all permutations $\langle \gamma \rangle$ of $V(\sigma_0)$. The group of projectivities is a subgroup of the symmetric group $\text{Sym}(V(\sigma_0))$ on the vertices of σ_0 .

The projectivities along null-homotopic closed facet paths based at σ_0 generate the subgroup $\Pi_0(K, \sigma_0)$ of $\Pi(K, \sigma_0)$, which is called the *reduced group of projectivities*. Finally, if K is strongly connected then $\Pi(K, \sigma_0)$ and $\Pi(K, \sigma'_0)$, respectively $\Pi_0(K, \sigma_0)$ and $\Pi_0(K, \sigma'_0)$, are isomorphic for any two facets $\sigma_0, \sigma'_0 \in K$. In this case we usually omit the base facet in the notation of the (reduced) group of projectivities, and write $\Pi(K) = \Pi(K, \sigma_0)$, respectively $\Pi_0(K) = \Pi_0(K, \sigma_0)$.

The odd subcomplex. Let K be locally strongly connected; in particular, K is pure. The link of a co-dimension 2-face f is a graph which is connected since K is locally strongly connected, and f is called *even* if the link $\text{lk}_K(f)$ of f is 2-colorable (i.e. bipartite as a graph), and *odd* otherwise. We define the *odd subcomplex* of K as all odd co-dimension 2-faces (together with their proper faces), and denote it by K_{odd} (or sometimes $\text{odd}(K)$).

Assume that K is pure and admits a $(d+1)$ -coloring of its graph $\Gamma(K)$, that is, we assign one color of a set of $d+1$ colors to each vertex of $\Gamma(K)$ such that the two vertices of any edge carry different colors. Observe that the $(d+1)$ -coloring of K is minimal with respect to the number of colors, and is unique up to renaming the colors if K is strongly connected. Simplicial complexes that are $(d+1)$ -colorable are called *foldable*, since a $(d+1)$ -coloring defines a non-degenerated simplicial map of K to the $(d+1)$ -simplex. In other places in the literature foldable simplicial complexes are sometimes called *balanced*.

Lemma 1.3. *The odd subcomplex of a foldable simplicial complex K is empty, and the group of projectivities $\Pi(K, \sigma_0)$ is trivial. In particular we have $\langle \gamma \rangle = \langle \delta \rangle$ for any two facet paths γ and δ from σ to τ for any two facets $\sigma, \tau \in K$.*

We leave the proof to the reader. As we will see in Theorem 1.4 the odd subcomplex is of interest in particular for its relation to $\Pi_0(K, \sigma_0)$ of a nice simplicial complex K .

Consider a geometric realization $|K|$ of K . To a given facet path $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$ in K we associate a (piecewise linear) path $|\gamma|$ in $|K|$ by connecting the barycenter of σ_i to the barycenters of $\sigma_i \cap \sigma_{i-1}$ and $\sigma_i \cap \sigma_{i+1}$ by a straight line for $1 \leq i < k$, and connecting the barycenters of σ_0

and $\sigma_0 \cap \sigma_1$, respectively σ_k and $\sigma_k \cap \sigma_{k-1}$. A projectivity *around* a co-dimension 2-face f is a projectivity along a facet path $\gamma \delta \gamma^-$, where δ is a closed facet path in $\text{st}_K(f)$ (based at some facet $\sigma \in \text{st}_K(f)$) such that $|\gamma|$ is homotopy equivalent to the boundary of a transversal disc around $|f| \subset |\text{st}_K(f)|$, and γ is a facet path from σ_0 to σ . The path $\gamma \delta \gamma^-$ is null-homotopic since K is locally strongly simply connected.

Theorem 1.4 (Izmestiev & Joswig [14, Theorem 3.2.2]). *The reduced group of projectivities $\Pi_0(K, \sigma_0)$ of a nice simplicial complex K is generated by projectivities around the odd co-dimension 2-faces. In particular, $\Pi_0(K, \sigma_0)$ is generated by transpositions.*

The fundamental group $\pi_1(|K| \setminus |K_{\text{odd}}|, y_0)$ of a nice simplicial complex K is generated by paths $|\gamma|$, where γ is a closed facet path based at σ_0 , and y_0 is the barycenter of σ_0 ; see [14, Proposition A.2.1]. Furthermore, due to Theorem 1.4 we have the group homomorphism

$$\mathfrak{h}_K : \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \rightarrow \Pi(K, \sigma_0) : [|\gamma|] \mapsto \langle \gamma \rangle,$$

where $[|\gamma|]$ is the homotopy class of the path $|\gamma|$ corresponding to a facet path γ .

The partial unfolding. Let K be a pure simplicial d -complex and set Σ as the set of all pairs $(|\sigma|, v)$, where $\sigma \in K$ is a facet and $v \in \sigma$ is a vertex. Thus each pair $(|\sigma|, v) \in \Sigma$ is a copy of the geometric simplex $|\sigma|$ labeled by one of its vertices. For neighboring facets σ and τ of K we define the equivalence relation \sim by attaching $(|\sigma|, v) \in \Sigma$ and $(|\tau|, w) \in \Sigma$ along their common ridge $|\sigma \cap \tau|$ if $\langle \sigma, \tau \rangle(v) = w$ holds. Now the *partial unfolding* \hat{K} is defined as the quotient space $\hat{K} = \Sigma / \sim$. The projection $p : \hat{K} \rightarrow K$ is given by the factorization of the map $\Sigma \rightarrow K : (|\sigma|, v) \mapsto \sigma$; see Figure 2.

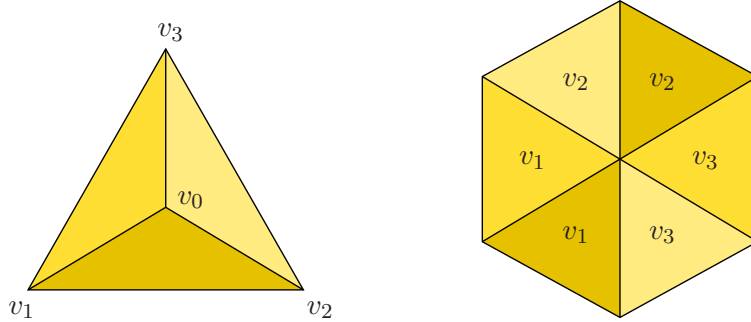


FIGURE 2. The starred triangle and its partial unfolding. The complex on the right is the non-trivial connected component of the partial unfolding, indicated by the labeling of the facets by the vertices v_1 , v_2 , and v_3 . The second connected component is a copy of the starred triangle with all facets labeled v_0 ; see also Example 1.1 for $k = 2$.

The partial unfolding of a strongly connected simplicial complex is not strongly connected in general. We denote by $\hat{K}_{(|\sigma|, v)}$ the connected component containing the labeled facet $(|\sigma|, v)$. Clearly, $\hat{K}_{(|\sigma|, v)} = \hat{K}_{(|\tau|, w)}$ holds if and only if there is a facet path γ from σ to τ in K with $\langle \gamma \rangle(v) = w$. It follows that the connected components of \hat{K} correspond to the orbits of the action of $\Pi(K, \sigma_0)$ on $V(\sigma_0)$. Note that each connected component of the partial unfolding is strongly connected and locally strongly connected [24, Satz 3.2.2]. Therefore we do not distinguish between connected and strongly connected components of the partial unfolding.

The problem that the partial unfolding \hat{K} may not be a simplicial complex can be addressed in several ways. Izmestiev & Joswig [14] suggest barycentric subdivision of \hat{K} , or anti-prismatic subdivision of K . A more efficient solution (with respect to the size of the resulting triangulations) is given in [24].

1.4. The partial unfolding as a branched cover. As preliminaries to this section we state two theorems by Fox [4] and Izmistiev & Joswig [14]. Together they imply that under the “usual connectivity assumptions” the partial unfolding of a simplicial complex is indeed a branched cover as suggested in the heading of this section.

Theorem 1.5 (Izmestiev & Joswig [14, Theorem 3.3.2]). *Let K be a nice simplicial complex. Then the restriction of $p : \widehat{K} \rightarrow K$ to the preimage of the complement of the odd subcomplex is a simple covering.*

Theorem 1.6 (Fox [4, p. 251]; Izmistiev & Joswig [14, Proposition 4.1.2]). *Let J and K be nice simplicial complexes and let $f : J \rightarrow K$ be a simplicial map. Then the map f is a simplicial branched cover if and only if*

$$\text{codim } K_{\text{sing}} \geq 2.$$

Since the partial unfolding of a nice simplicial complex is nice Corollary 1.7 follows immediately.

Corollary 1.7. *Let K be a nice simplicial complex. The projection $p : \widehat{K} \rightarrow K$ is a simple branched cover with the odd subcomplex K_{odd} as its branching set.*

For the rest of this section let K be a nice simplicial complex and let y_0 be the barycenter of a fixed facet $\sigma_0 \in K$. The projection $p : \widehat{K} \rightarrow K$ is a branched cover with $K_{\text{sing}} = K_{\text{odd}}$ by Corollary 1.7, and Izmistiev & Joswig [14] proved that there is a bijection $\iota : p^{-1}(y_0) \rightarrow V(\sigma_0)$ that induces a group isomorphism $\iota_* : \text{Sym}(p^{-1}(y_0)) \rightarrow \text{Sym}(V(\sigma_0))$ such that the following Diagram (1) commutes.

$$(1) \quad \begin{array}{ccc} \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) & & \\ \downarrow \mathfrak{m}_p & \searrow \mathfrak{h}_K & \\ \mathfrak{M}_p & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) \end{array}$$

Let $r : X \rightarrow Y$ be a branched cover and assume that there is a homomorphism of pairs $\varphi : (Y, Y_{\text{sing}}) \rightarrow (|K|, |K_{\text{odd}}|)$, that is, $\varphi : Y \rightarrow |K|$ is a homomorphism with $\varphi(Y_{\text{sing}}) = |K_{\text{odd}}|$. Then Theorem 1.8 gives sufficient conditions for $p : \widehat{K} \rightarrow K$ and $r : X \rightarrow Y$ to be equivalent branched covers. It is the key tool in the proof of the main Theorem 2.1 in Section 2.

Theorem 1.8. *Let K be a nice simplicial complex and let $r : X \rightarrow Y$ be a (simple) branched cover. Further assume that there is a homomorphism of pairs $\varphi : (Y, Y_{\text{sing}}) \rightarrow (|K|, |K_{\text{odd}}|)$, and let $y_0 \in Y$ be a point such that $\varphi(y_0)$ is the barycenter of $|\sigma_0|$ for some facet $\sigma_0 \in K$. The branched covers $p : \widehat{K} \rightarrow K$ and $r : X \rightarrow Y$ are equivalent if there is a bijection $\iota : r^{-1}(y_0) \rightarrow V(\sigma_0)$ that induces a group isomorphism $\iota_* : \mathfrak{M}_r \rightarrow \Pi(K, \sigma_0)$ such that the diagram*

$$(2) \quad \begin{array}{ccc} \pi_1(Y \setminus Y_{\text{sing}}, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, \varphi(y_0)) \\ \downarrow \mathfrak{m}_r & & \downarrow \mathfrak{h}_K \\ \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) \end{array}$$

commutes. In particular, we have $\widehat{K} \cong X$.

Proof. Corollary 1.7 ensures that $p : \widehat{K} \rightarrow K$ is indeed a branched cover, and commutativity of Diagram (1) and Diagram (2) proves commutativity of their composition:

$$\begin{array}{ccccc} \pi_1(Y \setminus Y_{\text{sing}}, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, \varphi(y_0)) & & \\ \downarrow \mathfrak{m}_r & & \downarrow \mathfrak{h}_K & \searrow \mathfrak{m}_p & \\ \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) & \xleftarrow{\iota_*} & \mathfrak{M}_p \end{array}$$

Theorem 1.2 completes the proof. □

2. CONSTRUCTING BRANCHED COVERS

Throughout this section let $r : X \rightarrow \mathbb{S}^d$ be a branched cover of the d -sphere with branching set F . The main objective is to give a large class of branched covers r , such that there is a combinatorial sphere S with $p : \widehat{S} \rightarrow S$ equivalent to r as a branched cover. In particular this implies the existence of a homeomorphism of pairs $\varphi : (\mathbb{S}^d, F) \rightarrow (|S|, |S_{\text{odd}}|)$. Note that by the nature of the partial unfolding and the projection $p : \widehat{S} \rightarrow S$ any branched cover r equivalent to p has to be simple and $(d+1)$ -fold. A theorem similar to Theorem 2.1 may easily be formulated for branched covers of d -balls.

Recall that we associate to a facet path γ in S the (realized) path $|\gamma|$ in $|S|$, and that the square brackets denote the homotopy class of a closed path. Thus we write $\mathbf{m}_r([\varphi^{-1}(|\gamma|)])$ for the image of an element in $\pi_1(\mathbb{S}^d \setminus F, y_0)$ represented by the closed path $\varphi^{-1}(|\gamma|)$, which in turn is obtained from a closed facet path γ based at some facet $\sigma_0 \in S$ with barycenter $\varphi(y_0)$ by first considering its realization $|\gamma|$ and then its preimage under φ .

Theorem 2.1. *For $d \geq 2$ let $r : X \rightarrow \mathbb{S}^d$ be a $(d+1)$ -fold, simple branched cover of the d -sphere, and assume that the branching set F of r can be embedded via a homeomorphism $\varphi : \mathbb{S}^d \rightarrow |S'|$ into the co-dimension 2-skeleton of a shellable simplicial d -sphere S' . Then there is a shellable simplicial d -sphere S , such that $p : \widehat{S} \rightarrow S$ is a branched cover equivalent to r . Further more, the d -sphere S can be obtained from S' by a finite series of stellar subdivision of edges. If S' is the boundary of a simplicial $(d+1)$ -polytope then also S is the boundary of a simplicial $(d+1)$ -polytope.*

To make the proof of Theorem 2.1 more digestible we first give the (algorithmical) back-bone of the proof and defer some of the more technical aspects to the Lemmas 2.2, 2.3, and 2.4. Fix a point $y_0 \in \mathbb{S}^d \setminus F$ and we may assume $\varphi(y_0)$ to be the barycenter of some facet $\sigma_0 \in S'$ and $|\sigma_0| \cap \varphi(F) = \emptyset$ to hold. Further fix a bijection ι between the preimage $\{x_0, x_1, \dots, x_d\} = r^{-1}(y_0)$ of y_0 and the vertices of σ_0 , and color the vertices of σ_0 via ι by the elements in $r^{-1}(y_0)$.

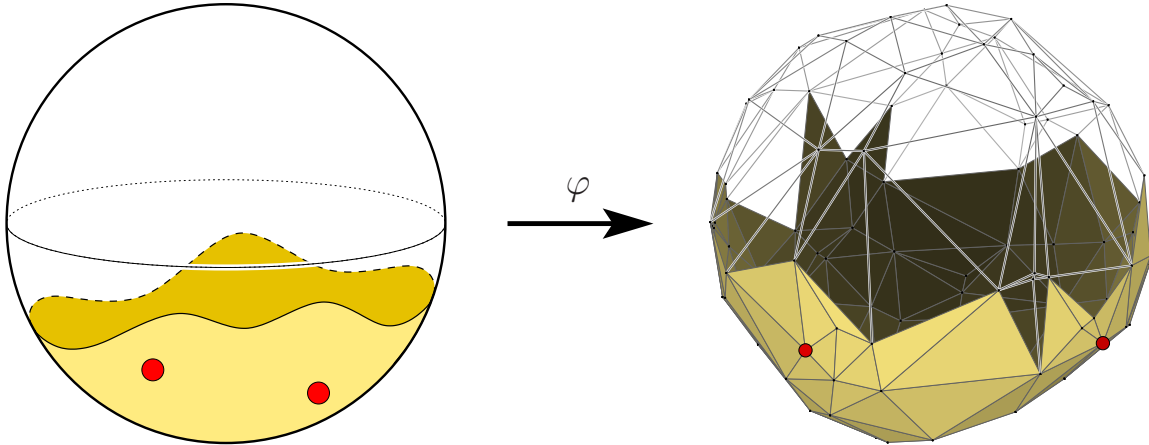


FIGURE 3. The base space of the branched cover $r : X \rightarrow \mathbb{S}^2$ (left) and a polytopal 2-sphere S_i with marked beginning $(\sigma_j)_{0 \leq j \leq l_i}$ of a shelling (right). On the left the preimage of $D_i = \bigcup_{0 \leq j \leq l_i} \sigma_j$ under the homomorphism $\varphi : \mathbb{S}^2 \rightarrow |S_i|$ is shaded and the branching set is marked. The odd subcomplex of D_i is marked on the right. The branched covers $r : X \rightarrow \mathbb{S}^2$ (restricted to $\varphi^{-1}(|D_i|)$) and $\widehat{D}_i \rightarrow D_i$ are equivalent.

The d -sphere S is constructed in a finite series ($S' = S_0, S_1, \dots, S_l = S$) of shellable d -spheres, and each d -sphere S_i comes with a shelling of its facet with marked beginning $(\sigma_{i,0}, \sigma_{i,1}, \dots, \sigma_{i,l_i})$. The complex S_{i+1} is obtained from S_i by (possibly) subdividing σ_{i,l_i+1} in a finite series of stellar subdivisions of edges not contained in any $\sigma_{i,j}$ for $0 \leq j \leq l_i$. Thus we may choose the shelling of S_{i+1} such that it extends $(\sigma_{i,0}, \sigma_{i,1}, \dots, \sigma_{i,l_i})$ and we denote the marked beginning of the shelling of S_i simply by $(\sigma_0, \sigma_1, \dots, \sigma_{l_i})$.

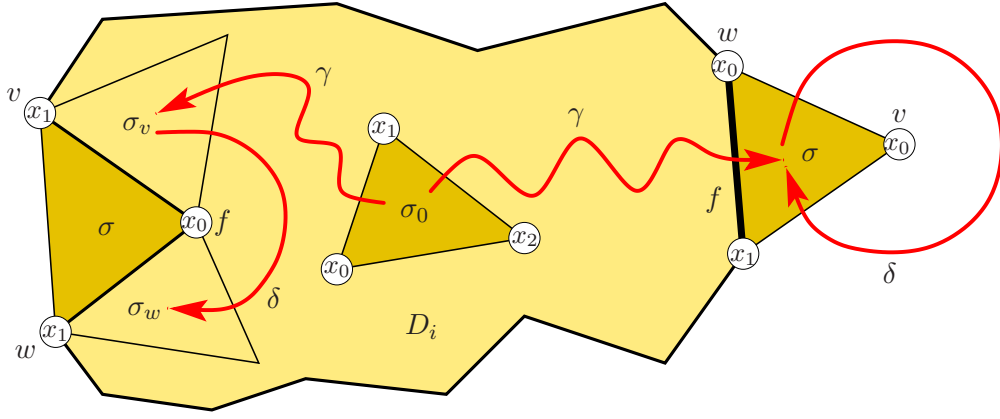


FIGURE 4. Case (i): The 2-ball D_i with the facet σ_0 colored via ι by the preimage $\{x_0, x_1, x_2\}$ of y_0 and induced coloring of the ridge f on the right hand side of the figure. The vertex v is colored x_0 if any element of \mathfrak{M}_r , corresponding via $m_r \circ \varphi^{-1}$ to a facet path of the form $\gamma \delta \gamma^{-1}$ maps x_0 to itself. Case (ii): The induced coloring of the co-dimension 2-face f and the vertices v and w on the left. The edge $\{v, w\}$ is subdivided if the facet path $\gamma \delta (\sigma_w, \sigma, \sigma_v) \gamma^{-1}$ corresponds via $m_r \circ \varphi^{-1}$ to the identity in \mathfrak{M}_r .

Let $D_i = \bigcup_{0 \leq j \leq l_i} \sigma_j$ then the main idea of the proof of Theorem 2.1 is to construct S_i such that the branched covers $r : X \rightarrow \mathbb{S}^d$ (restricted to $\varphi^{-1}(|D_i|)$) and $\widehat{D}_i \rightarrow D_i$ are equivalent. To this end we prove that φ restricted to $\varphi^{-1}(|D_i|)$ is a homomorphism of pairs $(\varphi^{-1}(|D_i|), F \cap \varphi^{-1}(|D_i|)) \rightarrow (|D_i|, |\text{odd}(D_i)|)$ and that the following Diagram (3) commutes; see Figure 3.

$$(3) \quad \begin{array}{ccc} \pi_1(\varphi^{-1}(|D_i|) \setminus F, y_0) & \xrightarrow{\varphi_*} & \pi_1(|D_i| \setminus |\text{odd}(D_i)|, \varphi(y_0)) \\ \downarrow m_r & & \downarrow h_{D_i} \\ \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(D_i, \sigma_0) \end{array}$$

Commutativity of Diagram (3) is obtained by ensuring that for each closed facet path γ in D_i (which is not a facet path in D_{i-1}) the projectivity $\langle \gamma \rangle$ acts on $V(\sigma_0)$ as $m_r([\varphi^{-1}(|\gamma|)])$ acts on $r^{-1}(y_0)$.

The pair $(S_{i+1}, (\sigma_j)_{0 \leq j \leq l_{i+1}})$ is constructed from the pair $(S_i, (\sigma_j)_{0 \leq j \leq l_i})$ as follows. Let $\sigma = \sigma_{l_{i+1}}$ be the first facet in the shelling of S_i not contained in D_i , let γ be a facet path in $D_i \cup \sigma$ from σ_0 to σ , and let $f \subset \sigma$ be a face. Further let $H_{f, \gamma}$ be the subgroup of \mathfrak{M}_r which is induced via m_r by all elements of $\pi_1(\mathbb{S}^d \setminus F, y_0)$ of the form $[\varphi^{-1}(|\gamma \delta \gamma^{-1}|)]$, where δ is any closed facet path in $\text{st}_{S_i}(f)$ based at σ . The subgroup $H_{f, \gamma}$ has at least $\dim(f) + 1$ trivial orbits, namely, the orbits corresponding to the vertices of f , and for $g \subset f$ we have that the set of trivial orbits of $H_{f, \gamma}$ contains the trivial orbits of $H_{g, \gamma}$. We consider the following three case:

- (i) The intersection $\sigma \cap D_i$ is a ridge f . Let γ be a facet path in $D_i \cup \sigma$ from σ_0 to σ , and color σ (and hence f) by the coloring induced along γ by the fixed coloring of σ_0 . Now keep the coloring of f , but change the color of the remaining vertex $v = \sigma \setminus f$ to any trivial orbit of $H_{v, \gamma}$; see Figure 4 (right).
- (ii) The intersection $\sigma \cap D_i$ equals two ridges $f \cup v$ and $f \cup w$ with a common co-dimension 2-face f . Let $\sigma_v \in D_i$ be the facet intersecting σ in $f \cup v$, let $\sigma_w \in D_i$ be the facet intersecting σ in $f \cup w$, and choose facet paths γ from σ_0 to σ_v in D_i and δ from σ_v to σ_w in $\text{st}_{D_i}(f)$. The fixed coloring of σ_0 induces along γ , respectively $\gamma \delta$, colorings on $f \cup v$ and $f \cup w$, and the colorings coincide on f . Now we change the color of w according to $m_r([\varphi^{-1}(|\gamma \delta (\sigma_w, \sigma, \sigma_v) \gamma^{-1}|)])$, which is either a transposition (changing the color of w) or the identity; see Figure 4 (left).

- (iii) Otherwise set $S_{i+1} = S_i$ and let $(\sigma_0, \sigma_1, \dots, \sigma_{l_i}, \sigma)$ be the marked beginning of a shelling of S_{i+1} .

We obtained a (possibly inconsistent) coloring of the vertices of σ in the cases (i) and (ii). Note that the coloring of σ induces a consistent coloring on $D_i \cap \sigma$, and that there is at most one *conflicting edge* $\{v, w\}$, that is, v and w are colored the same. A consistently colored subdivision of σ is constructed in at most $d - 1$ subdivisions of σ with exactly one conflicting edge e each, where each subdivision is obtained from the previous one by stellar subdividing e : Let $f_e \subset \sigma$ be the unique minimal face such that $|e| \subset |f|$ holds and denote by C_e the set of trivial orbits of $H_{f_e, \gamma}$. Now color the new vertex v_e with an element of C_e which is not the color of any vertex $v_{e'}$ subdividing an edge e' with $f_{e'} \subset f_e$. Note that C_e is the entire preimage $r^{-1}(y_0)$ if f_e is a co-dimension 1-face, and that C_e has at least one element distinct from the colors of all $v_{e'}$ for $f_{e'} \subset f_e$. If C_e contains the one color $x \in r^{-1}(y_0)$ not used in the coloring of σ , color v_e by x and terminate the subdivision process.

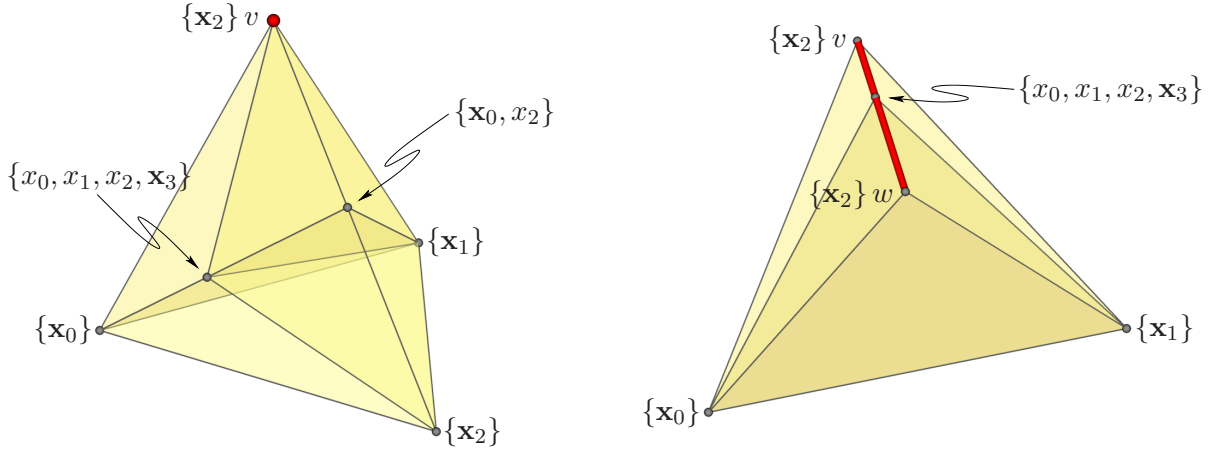


FIGURE 5. Coloring of the vertices of the refinement of σ in case (i) (on the left) and case (ii) (on the right). The minimal free face v , respectively $\{v, w\}$, is marked. Each vertex v_e is labeled by the trivial orbits of $H_{f_e, \gamma}$ and the vertex color is printed bold.

This completes the construction of S_{i+1} in the cases (i) and (ii), and we define the marked beginning of a shelling of S_{i+1} by $(\sigma_0, \sigma_1, \dots, \sigma_{l_i})$ followed by the facets of the refinement of σ in an appropriate order.

It remains to prove that the algorithm described above terminates and that $p : \widehat{S} \rightarrow S$ is a branched cover equivalent to $r : X \rightarrow \mathbb{S}^d$. Since S is shellable and hence nice, p is a branched cover by see Corollary 1.7. The following Lemmas 2.2 and 2.3 prove the equivalence of p and r , while termination of the construction above is provided by Lemma 2.4.

Lemma 2.2. *The branched covers $p : \widehat{S} \rightarrow S$ and $r : X \rightarrow \mathbb{S}^d$ are equivalent.*

Proof. In order to show the equivalence of the branched covers p and r we prove by induction that the following holds for $0 \leq i \leq l$:

- (I) For any closed facet path γ based at σ_0 in D_i we have

$$\langle \gamma \rangle = \iota_* \circ \mathbf{m}_r([\varphi^{-1}(|\gamma|)]).$$

- (II) Let $v \in D_i$ be a vertex, and let γ be a facet path in D_i from σ_0 to a facet σ containing v . Then the color induced on v along γ by the fixed coloring of σ_0 is a trivial orbit of $H_{v, \gamma}$.

We remark that (I) implies that φ restricted to $\varphi^{-1}(|D_i|)$ is a homomorphism of pairs $(\varphi^{-1}(|D_i|), F \cap \varphi^{-1}(|D_i|)) \rightarrow (|D_i|, |\text{odd}(D_i)|)$ and that the Diagram (3) commutes. Finally, (I) and (II) are met for the pair $(S_0, D_0) = (S', \sigma_0)$, and commutativity of Diagram (3) proves the equivalence of $r : X \rightarrow \mathbb{S}^d$ and $p : \widehat{S} \rightarrow S$ for $i = l$; see Theorem 1.8.

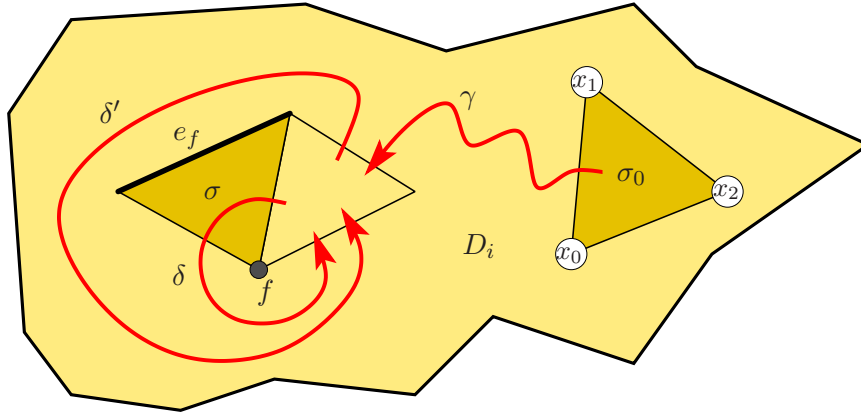


FIGURE 6. Case (iii): The paths γ , δ , and δ' if the corresponding edge e_f of a co-dimension 2-face f is non-free.

We show that (I) and (II) hold for the pair (S_{i+1}, D_{i+1}) provided they hold for the pair (S_i, D_i) . Recall that we denote the first facet σ_{i+1} of the shelling of S_i not contained in D_i by σ . The simplicial complex D_i is contractible and hence $\Pi_0(D_i, \sigma_0) = \Pi(D_i, \sigma_0)$ is generated by closed facet paths around (odd) co-dimension 2-faces by Theorem 1.4. Thus it suffices to verify (I) for closed facet paths around (interior) co-dimension 2-faces by examining the three cases (i), (ii), and (iii).

- (i) The intersection $\sigma \cap D_i$ is a ridge f . New interior co-dimension 2-faces in D_{i+1} arise only in the refinement of σ , which is foldable by construction. Since $\varphi(F)$ does not intersect the interior of $|\sigma|$, any facet path around a new interior co-dimension 2-face corresponds to the identity of \mathfrak{M}_r and (I) holds by Lemma 1.3.
- (ii) The intersection $\sigma \cap D_i$ equals two ridges $f \cup v$ and $f \cup w$ with a common co-dimension 2-face f . By induction hypothesis (II) holds for the vertices of f in D_i and thus (I) follows for the new interior co-dimension 2-face f of D_{i+1} by construction. As for any new interior co-dimension 2-face in the refinement of σ , (I) holds (as in case (i)) since the refinement is foldable and $\varphi(F)$ does not intersect the interior of $|\sigma|$.
- (iii) Otherwise there is no co-dimension 2-faces $f \subset \sigma$ with a free corresponding edge $e_f = \sigma \setminus f$ and (I) follows from Lemma 2.3.

Having established (I), it suffices to verify (II) for a single facet path γ in D_{i+1} from σ_0 to any facet containing a given vertex v . Thus (II) holds by choice of color for any vertex added to D_i in the construction of the pair (S_{i+1}, D_{i+1}) . \square

Lemma 2.3. *If $f \in \sigma$ is a co-dimension 2-face with a non-free corresponding edge $e_f = \sigma \setminus f$, then (I) holds for any closed facet path based at σ_0 around f in D_{i+1} .*

Proof. Let $\gamma \delta \gamma^{-1}$ be a closed facet path based at σ_0 around f in D_{i+1} , where δ is a closed path around f in $\text{st}_{D_{i+1}}(f)$. Since $\{v, w\} = e_f$ is a non-free edge, there is a facet path δ' in D_i with $|\delta'|$ homotopy equivalent to $|\{f_e, f \cup v, f \cup w\}|$ in $|D_i| \setminus |\text{odd}(D_i)|$, and we assume δ and δ' to have the same orientation; see Figure 6. Note that the complex $\{f_e, f \cup v, f \cup w\}$ itself is homotopy equivalent to \mathbb{S}^1 .

W.l.o.g. let $m_r([\varphi^{-1}(|\gamma \delta \gamma^{-1}|)])$ either be the identity or the transposition $(x_0, x_1) \in \mathfrak{M}_r$. Each transposition (x_i, x_j) , for $i \neq j$, appears at most once in the (unique) reduced representation of the element $a = m_r([\varphi^{-1}(|A|)]) \in \mathfrak{M}_r$ corresponding to the facet path $A = \gamma \delta' \gamma^{-1}$, since A is composed from facet paths around co-dimension 2-faces of σ . Let $b = m_r([\varphi^{-1}(|B|)]) \in \mathfrak{M}_r$ denote the element corresponding to the facet path $B = \gamma \delta' \delta^{-1} \gamma^{-1}$, then $a = (x_0, x_1) \circ b$ holds if and only if (x_0, x_1) is in the reduced representation of a , and we have $a = b$ otherwise. Since (I) holds for D_i and hence in particular for the facet path A , and with

$$A = \gamma \delta' \gamma^{-1} = \gamma \delta' \delta^{-1} \gamma^{-1} \gamma \delta \gamma^{-1} = B \gamma \delta \gamma^{-1},$$

we conclude that the projectivity along $\gamma\delta\gamma^{-1}$ is the identity on the vertices of σ_0 if and only if $\gamma\delta\gamma^{-1}$ corresponds via $\mathfrak{m}_r \circ \varphi^{-1}$ to the identity in \mathfrak{M}_r , and exchanges exactly the vertices colored x_0 and x_1 otherwise. \square

The following Lemma 2.4 proves termination of the construction of the shellable d -sphere S and completes the proof of Theorem 2.1.

Lemma 2.4. *The shellable d -sphere S is obtained by finitely many stellar subdivisions of edges.*

Proof. We prove that no facet will be subdivided more than a finite number of times in the construction of S . The facet σ_{l_i+i} is subdivided at most $d-1$ times in the construction S_{i+1} from S_i , and no facet in D_i is subdivided. The refinement of σ_{l_i+1} is added to D_i to define D_{i+1} and no facet in the refinement will be subdivided any further.

Problems may accrue since subdividing σ_{l_i+1} results in subdividing other facets (not in D_i) intersecting σ_{l_i+1} , and each facet of the refinement of an intersecting facet appears in the shelling, yet is not in D_{i+1} . Thus a facet might get subdivided over and over again.

For a face $f \in S'$ let $L_{f,i} \subset S_i$ denote the refinement of f in S_i . W.l.o.g. we may assume that the facets of the refinement $L_{\sigma,i}$ of any facet $\sigma \in S'$ appear consecutively in the shelling order of S_i . Let $\sigma \in S'$ be a fixed facet and let i_0 be the number such that S_{i_0} is the d -sphere with $\sigma_{l_{i_0}+1}$ is the facet of L_{σ,i_0} appearing first in the shelling order, that is, S_{i_0+1} is constructed by adding (a refinement) of the first facet of L_{σ,i_0} to D_{i_0} . Thus we obtain an induced coloring of the boundary vertices of L_{σ,i_0} which is consistent on $D_{i_0} \cap L_{\sigma,i_0}$ by construction. Since $\varphi(F)$ does not intersect the interior of $|L_{\sigma,i_0}|$ and by Lemma 1.3, it remains to prove that this coloring of $D_{i_0} \cap L_{\sigma,i_0}$ extends to a foldable refinement of L_{σ,i_0} obtained via a finite series of stellar subdivisions.

Observe that each facet of L_{σ,i_0} is the cone over a $(d-1)$ -simplex in the boundary of L_{σ,i_0} and that L_{σ,i_0} has no interior vertices: This is obviously true for $L_{\sigma,0} = \sigma$. For $1 \leq i \leq i_0$ let $\text{cone}(f)$ be a facet of $L_{\sigma,i-1}$ with f is a boundary $(d-1)$ -simplex. Now if $\text{cone}(f)$ is subdivided via stellarly subdividing an edge $e \in f$, both facets replacing $\text{cone}(f)$ are cones over boundary $(d-1)$ -simplices which in turn are obtained from f by replacing one vertex of e by the new vertex subdividing e .

We strengthen the statement above and claim that each facet of L_{σ,i_0} is the cone over a $(d-1)$ -simplex in $D_{i_0} \cap L_{\sigma,i_0}$. To this end note the trivial fact that if $e \in L_{g,i}$ is an edge of the subdivision of a boundary k -face $g \in \sigma$ and let $\{f_j\}_{1 \leq j \leq d-k}$ be the boundary $(d-1)$ -faces of σ with $g = \bigcap_{1 \leq j \leq d-k} f_j$, then there is a $(d-1)$ -simplex in each $L_{f_j,i}$ containing e . Thus if for some $i < i_0$ an edge e is subdivided when adding the simplex σ_{l_i+1} to D_i which intersects $L_{\sigma,i}$ in a low dimensional face, then at least one of the boundary $(d-1)$ -simplices of $L_{\sigma,i}$ containing e will be added to $D_{i'} \cap L_{\sigma,i'}$ at some point $i < i' \leq i_0$.

Returning to the consistent coloring of $D_{i_0} \cap L_{\sigma,i_0}$ we conclude that all vertices of L_{σ,i_0} are colored since there are no interior vertices, and that each facet $\text{cone}(f)$ of L_{σ,i_0} has at most one conflicting edge since the boundary $(d-1)$ -simplex $f \subset D_{i_0} \cap L_{\sigma,i_0}$ is consistently colored. Hence $\text{st}_{L_{\sigma,i_0}}(e)$ of an conflicting edge e does not contain any other conflicting edges and we consider $\text{st}_{L_{\sigma,i_0}}(e)$ independently.

Now $\text{st}_{L_{\sigma,i_0}}(e)$ is subdivided only finitely many times since $H_{v,\gamma}$ is trivial for any new vertex v (except for finitely many vertices in the boundary of $|\text{st}_{L_{\sigma,i_0}}(e)|$) and hence the construction (case (i) and (ii)) induces a linear order on the colors used to color the new vertices. \square

Remark 2.5. It appears as if the shellable d -sphere S may be constructed along a spanning tree of the dual graph $\Gamma^*(S')$ instead of a shelling, though the construction would become substantially more complicated. Using a spanning tree of $\Gamma^*(S')$ would eliminate the some how (to the theory of branched covers) alien concept of a shelling, and would allow for more general base spaces, e.g. PL d -manifolds.

Applying Theorem 2.1 to the results of Hilden [8] and Montesinos [17], Piergallini [21], and Iori & Piergallini [11] we obtain the following three corollaries.

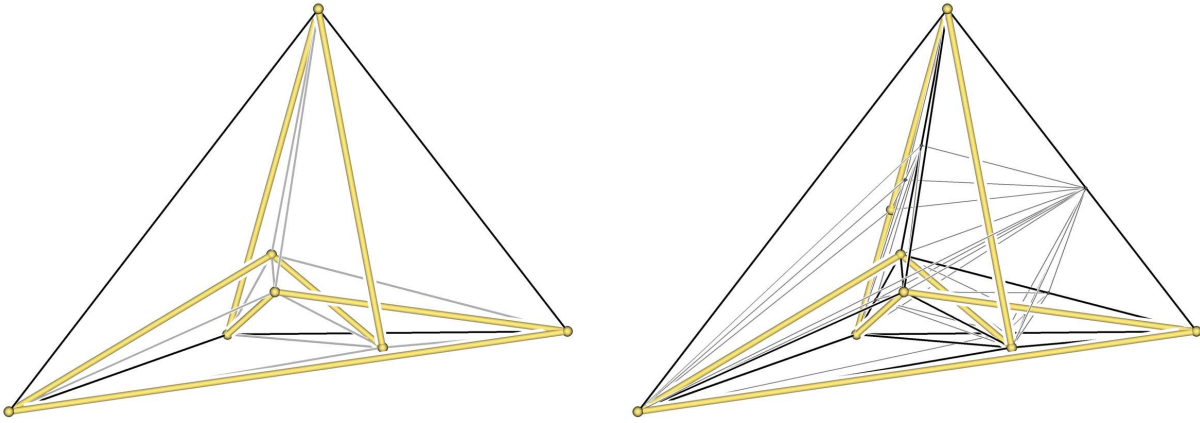


FIGURE 7. Constructing the trefoil as odd subcomplex of a 3-sphere S with $\Pi(S)$ isomorphic to the symmetric group on three elements. On the left the Schlegel diagram of the cyclic 4-polytope $C_{4,7}$ on seven vertices with the trefoil embedded in the 1-skeleton. On the right S as a subdivision of the Schlegel diagram after stellarily subdividing eight edges of $C_{4,7}$. The odd subcomplex is marked and the f -vector of S reads $(15, 63, 96, 48)$; watch [25].

Corollary 2.6. *Let $d = 2$ or $d = 3$. For every closed oriented d -manifold M there is a polytopal d -sphere S such that one of the connected components \hat{S} of the partial unfolding of S is a combinatorial d -manifold homeomorphic to M . The projection $\hat{S} \rightarrow S$ is a simple d -fold branched cover branched over finitely many points for $d = 2$, respectively a link for $d = 3$.*

Corollary 2.7. *For every closed oriented PL 4-manifold M there is a polytopal 4-sphere S such that one of the connected components \hat{S} of the partial unfolding of S is a combinatorial 4-manifold PL-homeomorphic to M . The projection $\hat{S} \rightarrow S$ is a simple 4-fold branched cover branched over a PL-surface with a finite number of cusp and node singularities.*

Corollary 2.8. *For every closed oriented PL 4-manifold M there is a polytopal 4-sphere S such that the partial unfolding \hat{S} of S is a combinatorial 4-manifold PL-homeomorphic to M . The projection $\hat{S} \rightarrow S$ is a simple 5-fold branched cover branched over a locally flat PL-surface.*

A weaker version of Corollary 2.6 was already established by Izmistiev & Joswig [14] and later by Hilden, Montesinos-Amilibia, Tejada & Toro [9]. A weaker version of Corollary 2.7 can be found in [26].

Stellar subdivision of an edge $e \in S$ of a combinatorial d -manifold S changes the parity of the co-dimension 2-faces in $\text{lk}_S(e)$. Since the link of an edge of S is a (combinatorial) $(d-2)$ -sphere, we obtain the following Corollary 2.9. A topological proof for arbitrary simple branched covers is available by Izmistiev [13].

Corollary 2.9. *The branching set of a branched cover $r : X \rightarrow \mathbb{S}^d$ as described in Theorem 2.1 is the symmetric difference of finitely many $(d-2)$ -spheres.*

We conclude this section by a remark and a conjecture as to which branched covers $r : X \rightarrow \mathbb{S}^d$ may be obtained via the method presented above. In other words, which branching sets can be embedded via a homomorphism $\varphi : \mathbb{S}^d \rightarrow |S'|$ into the co-dimension 2-skeleton of a shellable simplicial d -sphere S' .

Remark 2.10. For $d \geq 6$ there are branching sets non-embeddable into the co-dimension 2-skeleton of a shellable simplicial d -sphere: Freedman & Quinn [5] constructed a 4-manifold which does not have a triangulation as a combinatorial manifold. In fact, there are 4-manifolds which can not be triangulated at all [16, p. 9].

The branching set of a branched cover $r : X \rightarrow \mathbb{S}^d$ for $d \leq 5$ is at most 3-dimensional and since there is no difference in between PL and non-PL topology up to dimension three, we conjecture the following.

Conjecture 2.11. For $d \leq 5$ every branched cover $r : X \rightarrow \mathbb{S}^d$ can be obtained via the partial unfolding of some polytopal d -sphere.

3. EXTENDING TRIANGULATIONS

A first assault on how to extend triangulation and coloring is by Goodman & Onishi [7], who proved that a 4-colorable triangulation of the 2-sphere may be extended to a 4-colorable triangulation of the 3-ball. Their result was improved independently by Izmistiev [12] and [24] to arbitrary dimensions. Here we generalize the construction to arbitrary simplicial complexes with k -colored subcomplexes.

Theorem 3.1. *Given a simplicial d -complex K and a k -colored induced subcomplex L , then there is a finite series of stellar subdivisions of edges, such that the resulting simplicial complex K' has a $\max\{k, d+1\}$ -coloring, K' contains L as an induced subcomplex, and the $\max\{k, d+1\}$ -coloring of K' induces the original k -coloring on L .*

Proof. We may assume K to be pure. Let $K_0 = K$ and assign 0 to all vertices not in L . For $1 \leq i \leq d$ we obtain the simplicial complex K_i from K_{i-1} by stellar subdividing all conflicting edges with both vertices colored $i-1$ in an arbitrary order. The new vertices are colored i . We prove by induction that for $0 \leq j \leq i-1$ and each facet $\sigma \in K_i$ there is exactly one vertex $v_j \in \sigma$ colored j . The assumption holds for K_0 and completes the proof for $K' = K_d$. Note that since L is properly colored, no edges in L are subdivided and L is an induced subcomplex of any K_i for $0 \leq i \leq d$.

To prove the induction hypothesis for K_i , we again use an inductive argument: Let σ be a facet of a subdivision of K_{i-1} produced in the making of K_i . Assume that each color less than $i-1$ appears exactly once in σ , and let $l \geq 2$ be the number of $(i-1)$ -colored vertices of σ . This assumption clearly holds for any facet of K_{i-1} for some $l \leq d-i+2$. After subdividing an $(i-1)$ -colored conflicting edge of σ and assigning the color i to the new vertex, each of the two new facets has $l-1$ vertices colored $i-1$, and each color less than $i-1$ appears exactly once. Thus any facet of K_{i-1} has to be subdivided into at most 2^{d-i+1} simplices in order for K_i to meet the induction hypothesis. \square

Izmestiev gives a result similar to Theorem 3.1 in [12], but the following Remark 3.2 points out the advantage of using only stellar subdivisions of edges.

Remark 3.2. Since only stellar subdivisions of edges are used to construct K' from K all properties invariant under these subdivisions are preserved, e.g. polytopality, regularity, shellability, and others. In the case that L is not induced, stellarly subdivide all edges $\{v, w\} \in K \setminus L$ with $v, w \in L$. In order to obtain a small triangulation, one can try to (greedily) $(d+1)$ -color a (large) foldable subcomplex first.

Corollary 3.3. *The odd subcomplex of a closed combinatorial d -manifold is the symmetric difference of finitely many $(d-2)$ -spheres.*

Corollary 3.4. *Given a k -colored simplicial $(d-1)$ -sphere S , then there is a simplicial d -ball D with boundary equal to S , such that there is a $\max\{k, d+1\}$ -coloring of D which induces the original k -coloring on S . The d -ball D is obtained from $\text{cone}(S)$ by a finite series of stellar subdivision of edges. In particular D is a combinatorial d -ball if S is a combinatorial $(d-1)$ -sphere, shellable if S is shellable, and regular if S is polytopal; see Figure 8.*

Remark 3.5. Similar results as Corollary 3.4 may easily be obtained for partial triangulations of CW-complex and relative handle-body decomposition of PL-manifold (with boundary).

The partial unfoldings of two homeomorphic simplicial complexes K and K' need not to be homeomorphic in general. We present a notion of equivalence of simplicial complexes which agrees with their unfolding behavior, that is, we give sufficient criteria such that if $p : \widehat{K} \rightarrow K$ and $p' : \widehat{K'} \rightarrow K'$ are branched covers, then p and p' are equivalent.

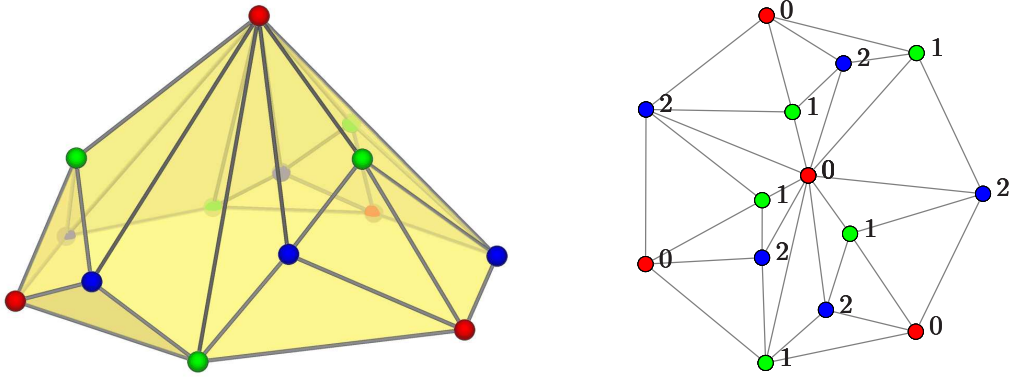


FIGURE 8. Convex hull of the extended triangulation of a 3-colored 7-gon and its Schlegel diagram.

Assume K and K' to be strongly connected and that the odd subcomplexes K_{odd} and K'_{odd} are *equivalent*, that is, there is a homeomorphism of pairs $\varphi : (|K|, |K_{\text{odd}}|) \rightarrow (|K'|, |K'_{\text{odd}}|)$. Let $\sigma_0 \in K$ be a facet, and y_0 the barycenter of σ_0 , and assume that the image $y'_0 = \varphi(y_0)$ is the barycenter of $|\sigma'_0|$ for some facet $\sigma'_0 \in K'$. Now K and K' are *color equivalent* if there is a bijection $\psi : V(\sigma_0) \rightarrow V(\sigma'_0)$, such that

$$(4) \quad \psi_* \circ \mathfrak{h}_K = \mathfrak{h}_{K'} \circ \varphi_*$$

holds, where the maps $\varphi_* : \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \rightarrow \pi_1(|K'| \setminus |K'_{\text{odd}}|, y'_0)$ and $\psi_* : \text{Sym}(V(\sigma_0)) \rightarrow \text{Sym}(V(\sigma'_0))$ are the group isomorphisms induced by φ and ψ , respectively.

Observe that this is indeed an equivalence relation. The name “color equivalent” suggests that the pairs (K, K_{odd}) and (K', K'_{odd}) are equivalent, and that the “colorings” of K_{odd} and K'_{odd} by the $\Pi(K)$ -action, respectively $\Pi(K')$ -action, of projectivities around odd faces are equivalent. Lemma 3.6 justifies this name.

Lemma 3.6. *Let K and K' be color equivalent nice simplicial complexes. Then the branched covers $p : \widehat{K} \rightarrow K$ and $p' : \widehat{K'} \rightarrow K'$ are equivalent.*

Proof. With the notation of Equation (4) we have that

$$\begin{array}{ccccc} \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K'| \setminus |K'_{\text{odd}}|, y'_0) \\ \swarrow m_p & \downarrow \mathfrak{h}_K & \downarrow \mathfrak{h}_{K'} & \searrow m_{p'} \\ \mathfrak{M}_p & \xrightarrow{\psi_*} & \Pi(K', \sigma'_0) & \xleftarrow{\iota'_*} & \mathfrak{M}_{p'} \end{array}$$

commutes, since the Diagram (1) commutes and Equation (4) holds. Theorem 1.2 completes the proof. \square

Proposition 3.7. *For every strongly connected simplicial complex K there is a simplicial complex K' obtained from a foldable simplicial complex via a finite series of stellar subdivision of edges, such that K and K' are color equivalent.*

Theorem 2.1 proves Proposition 3.7 above for shellable spheres. We will not prove the general case and only give a sketch of the construction for general K .

Let L be a foldable simplicial complex obtained from K via a finite series of stellar subdivisions according to Theorem 3.1, that is, there is a series $(K = K_0, K_1, \dots, K_l = L)$ where K_i is obtained from K_{i-1} by stellar subdividing a single edge $e_{i-1} \in K_{i-1}$. The idea is to reverse the effect of the stellar subdivisions by subdividing each edge e a second time in the reversed order, since stellar subdividing e twice yields the anti-prismatic subdivision of e (which does not alter the color equivalence class).

We construct K' from L inductively in a series $(L = L_l, L_{l-1}, \dots, L_0 = K')$ of simplicial complexes, where L_i is obtained from L_{i+1} by a finite series of stellar subdivision of edges. The

complexes L_i and K_i are color equivalent: For a facet path $(\sigma'_j)_{j \in J}$ in L_i associate the facet path $(\sigma_j)_{j \in J}$ in K_i , where σ_j is the unique facet, such that $|\sigma'_j|$ lies in $|\sigma_j|$.

We fix some notation in order to describe the construction of L_i from L_{i+1} . Subdividing the edge $e_i \in K_i$ in order to construct K_{i+1} replaces e_i by two edges in K_{i+1} , and we call one of these two edges e'_i . A facet in $\text{st}_{K_{i+1}}(e'_i)$ might get subdivided further in the process of constructing $K_{i+2}, K_{i+3}, \dots, K_l = L_l, L_{l-1}, \dots, L_{i+1}$, and we define $L_{e'_i}$ as the subcomplex of L_{i+1} which refines $\text{st}_{K_{i+1}}(e'_i)$.

Note that e'_i is an edge of $L_{e'_i}$, and that $L_{e'_i}$ and $\text{st}_{K_{i+1}}(e'_i)$ are color equivalent. It follows that the group of projectivities of $L_{e'_i}$ has at least two trivial orbits corresponding to the vertices of e'_i . Now L_i is obtained from L_{i+1} by stellarly subdividing all edges with vertices belonging to the same two trivial orbits as the vertices of e'_i .

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